Irradiance transport equation from geometrical optics considerations

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A general equation for the propagation of phase and irradiance is derived within the geometrical optics regime. The starting point is Poynting's theorem together with the representation of the Poynting vector in the eikonal approximation. It is shown that the irradiance transport equation is a particular case of a more general conservation equation and is valid in the paraxial regime. An analysis of the range of validity of the irradiance transport equation is performed.

Keywords: Irradiance transport equation; wavefront sensing; geometrical optics.

Se deriva una ecuación general para la propagación de la fase y la irradiancia dentro del régimen de la óptica geométrica. Se comienza con el teorema de Poynting junto con la representación del vector de Poynting en la aproximación eikonal. Se muestra que la ecuación de transporte de irradiancia es un caso particular de una ecuación de conservación más general y es valida en el régimen paraxial. Se realiza un análisis sobre el rango de validez de la ecuación de transporte de irradiancia.

Descriptores: Ecuación de transporte de irradiancia; sensor de frente de onda; óptica geométrica.

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1. Introduction

The irradiance transport equation (ITE) is usually obtained by assuming that the amplitude of light, which propagating mainly in one direction, approximately satisfies the parabolic equation [1,2]; this equation can be obtained from the Helmholtz equation [1]. The ITE is used in many schemes for recovering the phase from irradiance measurements, without using interferometric techniques [1-4]. In addition, this transport equation is the basis for curvature wavefront sensing methods [5,6]. The ITE relates the phase in a plane orthogonal to the optical axis to the rate of change of the beam intensity along the propagation direction, assuming a paraxial beam described by $\psi(x, y, z) = [I(x, y, z)]^{1/2} \exp[i\phi(x, y, z)]$, where $I(x, y, z)$ is the irradiance, and $\phi(x, y, z)$ is the phase. In terms of the wavefront $W(x, y, z)$, the phase is given by the relation $\phi(x, y, z) = (2\pi/\lambda)W(x, y, z) = kW(x, y, z)$, and the ITE states that

$$\nabla_T \cdot [I(x,y,z) \nabla_T W(x,y,z)] = -\frac{\partial}{\partial z} I(x,y,z), \quad (1)$$

where $\nabla_T \equiv \partial/\partial x \, \mathbf{x} + \partial/\partial y \, \mathbf{y}$ is the nabla operator in the $x - y$ plane, and z is the direction of the beam propagation. This equation was derived for light propagating mainly in one direction (*i.e.* the components of the wave vector $\mathbf{k}$ satisfy the condition $k_x, k_y \ll k_z$) [1-4]. In recent work [7], a more general transport equation applicable to inhomogeneous transparent media in the paraxial approximation has been obtained; here Eq. (1) is presented as a particular case of a homogeneous medium. More recently [8], the ITE has been derived from a subset of a basic equation used for finding the direction of the phase gradient; this basic equation comes from the imaginary part of the Helmholtz equation.

In general, the derivation of the ITE from physical optics considerations requires many approximations based on the Helmholtz equation [1,2], and apparently, in the mind of researchers it is a physical optics equation.

In this work the correlation between the phase and the irradiance will be developed from a geometrical optics point of view. It is important to point out that the derivation of the ITE using the representation of Poynting's vector in the eikonal approximation ($\lambda \to 0$) has not been previously developed. With this approach, we shall show that the ITE, which has been obtained based on the physical optics regime [1-8], is valid in the geometrical optics limit. For this, in Sec. 2, we establish the relation between the eikonal equation and Poynting's vector; then, considering the conservation law of the radiant energy, we obtain a general equation for irradiance and the phase from a geometrical optics point of view. Next, in Sec. 3, from the general equation relating intensity and wavefront (or phase), the ITE that gives the propagation of phase and intensity is obtained; here, the conditions that the wavefront must fulfill to satisfy the ITE are analyzed. Finally, in Sec. 4 conclusions are presented. It is important to point out that this work is part of the B.Sc. thesis of one of the authors [9], and here it is presented in an ordered way.

2. Correlation between phase and intensity

A wavefront $W(x, y)$, propagating along the z-direction experiences a change in both its phase and its amplitude (or irradiance). For an isotropic and a non-conducting medium, the geometrical light rays are defined as the orthogonal trajectories to the wavefront W; they coincide with the direction of the average Poynting vector. In a plane perpendicular to the wavefront propagation, a spatial distribution of the intensity is obtained due to the divergence and convergence of the rays

along the observation plane (Fig. 1); this intensity distribution gives information about the curvature of the wavefront.

In order to know the relationship between the phase and the intensity from the geometrical optics point of view, we consider a small portion of the wavefront δW and assume uniform intensity. In this case, the marginal rays M, the area δW, and a small area δA on the detection plane bound a volume Ω in space (Fig. 2). Inside this region, a unit vector **m** along the propagation direction of the averaged flux energy is given by

$$\mathbf{m} = \nabla W(x, y)/|\nabla W(x, y)| \qquad (2)$$

where **m** is a unit vector orthogonal to the wavefront δW, and $\nabla \equiv \partial/\partial x \, \mathbf{x} + \partial/\partial y \, \mathbf{y} + \partial/\partial z \, \mathbf{z}$, is the three-dimensional nabla operator.

In optics, the averaged Poynting vector is a quantity of great interest because its magnitude is a measure of the light intensity or irradiance, and its direction represents the direction of propagation of the light. For time-harmonic fields it can be shown that [10]

$$\langle \mathbf{S} \rangle = \frac{c}{8\pi} Re \left(\mathbf{E} \times \mathbf{H}^* \right). \qquad (3)$$

For an isotropic medium we have that $\langle \mathbf{S} \rangle = \langle S \rangle \mathbf{m}$, which establishes a relationship between the wavefront and the geometrical rays. From Poynting's theorem [10] it can be shown that the average energy flux passing through a closed surface
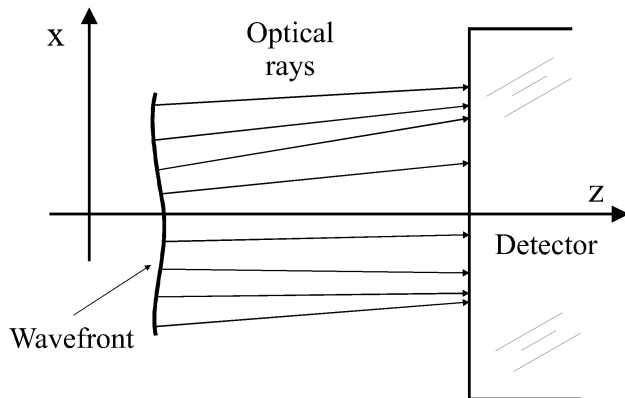

FIGURE 1. The intensity distribution on the detection plane gives information about the wavefront shape.
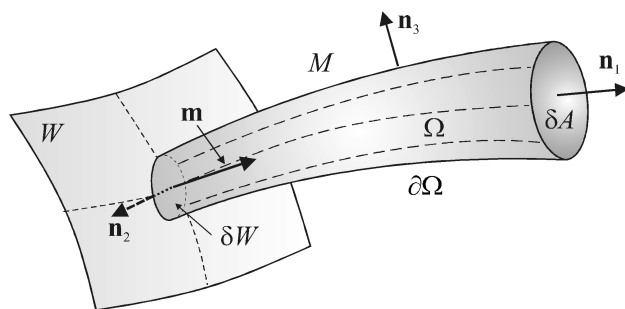

FIGURE 2. Radiant energy flux through the region Ω.

$\partial\Omega$, enclosing a volume Ωs is zero:

$$\int_{\partial\Omega} \langle \mathbf{S} \rangle \cdot \mathbf{n} \, ds = \sum_{i=1}^{3} \int_{\partial\Omega} \langle \mathbf{S} \rangle \, \mathbf{m} \cdot \mathbf{n}_i \, ds = 0, \qquad (4)$$

where $\partial\Omega = \{\delta W, M, \delta A\}$, and $\mathbf{n}_i$ $(i = 1,2,3)$ is the outward normal to $\partial\Omega$. Here the energy flowing in across the boundary surfaces of the volume per unit time is equal to the energy flowing out through the boundary surfaces of the volume per unit time. Or, equivalently,

$$\nabla \cdot \langle \mathbf{S} \rangle = 0. \qquad (5)$$

Thus, Eq. (5) shows that radiant energy is conserved. This is true for a non-conducting medium $(\sigma = 0)$ where no mechanical work is done; *i.e.*, it is valid for regions where there are no free charges or other singular points such as point sources or ray caustics.

On the other hand, it is well known that the eikonal equation [11,12], which describes the behavior of light rays in geometrical optics, is given by

$$(\nabla W)^2 = n^2, \qquad (6)$$

or, equivalently [11], by

$$\nabla W = n \, \mathbf{m}, \qquad (7)$$

where n is the refractive index, and **m** is the propagation unit vector given by Eq. (2). This expression relates the wavefront and the geometrical rays. Since the average Poynting vector is in the direction of the normal **m** to the geometrical wavefront, and its magnitude is equal to the product of the average energy density $\langle u \rangle$ and the velocity v, we have

$$\langle S \rangle = \mathbf{v} \langle u \rangle \, \mathbf{m}. \qquad (8)$$

This equation shows that the average energy density is propagated with velocity v along the **m**-direction within the accuracy of geometrical optics [11].

The intensity of the light is defined as the magnitude of the average Poynting vector. Therefore, we have from Eq. (8) that

$$I \equiv |\langle \mathbf{S} \rangle| = \mathbf{v} \langle u \rangle. \qquad (9)$$

Then, from Eqs. (6)-(9), the average Poynting vector can be written as

$$\langle \mathbf{S} \rangle = I \frac{\nabla W}{n}, \qquad (10)$$

which gives the relation between the eikonal equation and the average Poynting vector. Substituting Eq. (10) into the conservation law (5) leads to

$$\nabla \cdot \left(I \frac{\nabla W}{n} \right) = 0, \qquad (11)$$

which has been obtained completely within the geometrical optics limit.

Equation (11) is equivalent to that obtained from the general form of Poynting's vector in terms of amplitude and phase variables for a linearly polarized wave propagating

in an inhomogeneous transparent medium [7]. In addition, Eq. (11) can be obtained directly from the Helmholtz equation by rewriting the phase in terms of the optical path [8], *i.e.*, $\phi(x, y, z) = kW(x, y, z)$.

Assuming a homogenous medium, Eq. (11) may also be written as

$$I\nabla^2 W + \nabla W \cdot \nabla I = 0, \qquad (12)$$

where $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian operator. This formula gives us a general relationship between the phase and the intensity for a monochromatic wave propagating in a homogeneous non-conducting medium, and it is valid within the geometrical optics limit.

3. Propagation of phase and irradiance

To obtain the expression that gives the propagation of the phase and intensity (*i.e.* the explicit dependence with the z-direction coordinate), we rewrite Eq. (12) in terms of the nabla operator ∇_T in the $x - y$ plane:

$$I\nabla_T^2 W + I\frac{\partial^2}{\partial z^2}W + \nabla_T W \cdot \nabla_T I + \frac{\partial}{\partial z}I\frac{\partial}{\partial z}W = 0. \quad (13)$$

If the following conditions on the wavefront are satisfied

$$\left|\frac{\partial}{\partial z}W\right| \approx 1, \qquad (14)$$

and

$$\left|\frac{\partial^2}{\partial z^2}W\right| \ll \left|\nabla_T^2 W\right|, \qquad (15)$$

then Eq. (13) reduces to the ITE [Eq. (1)]:

$$I\nabla_T^2 W + \nabla_T W \cdot \nabla_T I + \frac{\partial}{\partial z}I = 0. \qquad (16)$$

This equation was derived by other authors from the physical optics point of view [1,2], but in this work we have obtained it entirely within the geometrical optics regime.

An important application of the solutions of the ITE [6,13] is in optical metrology. For the testing of optical surfaces or systems, the wavefront of a light beam reflected from or transmitted through an optical system can be determined from irradiance measurements through the ITE. In this case, we consider a flat detection surface, because in many cases optical metrology measurements are performed with a planar sensor; it can, however, be properly generalized. If we assume a plane sensor defined by the xy-plane, and the light propagates mainly in the z-direction, the phase of the beam can be written as

$$\phi(x, y, z) = \phi_T(x, y) + kz, \qquad (17)$$

where $\phi_T(x, y)$ denotes the phase in the sensor plane. Considering that the phase is given by $\phi = kW$, then, Eq. (17) fulfills the conditions in Eqs. (14)-(15), provided that $\nabla_T^2 W \neq 0$. In general, Eq. (17) for the phase of the beam can be satisfied in an almost collimated laser beam, or in the far-field of scattered light [8].

Another important case that fulfills the conditions (14)-(15) is the aberrated wavefront [14], $W(x, y, z)$, propagating in the z-direction. It is given by

$$W(x, y, z) = A\left(x^2 + y^2\right)^2 + By\left(x^2 + y^2\right)$$
$$+ C\left(x^2 + 3y^2\right) + D\left(x^2 + y^2\right)$$
$$+ Ey + Fx + G + z, \qquad (18)$$

where A is the spherical aberration, B coma, C astigmatism, D defocusing, E tilt in x, F tilt in y, and G piston coefficients; it is assumed that not all the aberration coefficients are null at the same time. According to the ITE [Eq. (1)], the aberrated wavefront experiences a change in both its intensity and its phase as it propagates along the z-axis.

If we consider a monochromatic plane wave that propagates in the direction given by $\mathbf{k} = k_x \mathbf{x} + k_y \mathbf{y} + k_z \mathbf{z}$, its wavefront is given by

$$W(x, y, z) = \frac{k_x}{k}x + \frac{k_y}{k}y + \frac{k_z}{k}z, \qquad (19)$$

where $k = |\mathbf{k}|$; then, in order to fulfill conditions (14)-(15), we must have that $k_z^2 \gg k_x^2 + k_y^2$. This means that the plane wave must propagate along (or almost along) the z-direction.

Finally, for a spherical wavefront,

$$W(x, y, z) = \left(x^2 + y^2 + z^2\right)^{1/2}; \qquad (20)$$

the conditions on the wavefront in Eq. (14)-(15) lead to $z^2 \gg x^2 + y^2$, and $z > 0$. Thus, the wave must propagate mainly in the z-direction and we have to stay very near the $z-$axis in the far field (paraxial approximation).

4. Conclusions

Based on Poynting's theorem together with Poynting's vector in the geometrical optics limit, an analytical general conservation expression has been obtained; this expression relates the phase and the intensity of a monochromatic field. In all the calculations we assume that the monochromatic field propagates in a homogeneous, isotropic, and non-conducting medium. The ITE has been obtained from the conservation equation for the same kind of medium, under conditions given by Eqs. (14)-(15). By examination of some examples we concluded that the wave must propagate mainly in the z-direction and the measurements must be done very near the z-axis. This means that the ITE is valid in the geometrical optics paraxial regime, where the optical path function W satisfies the eikonal equation.

Even though the ITE can be obtained from the physical optics point of view, in this work it has been shown that only geometrical optical considerations are sufficient to deduce it. This approach gives us an easy picture of the relation between the irradiance distribution and the wavefront. However, if we want to study the propagation of monochromatic fields

in an inhomogeneous media [7] or to know the phase gradient of monochromatic fields by means of three-dimensional intensity distributions [8], it is necessary to consider more general developments where the ITE is a particular case of them. Nevertheless, this approach will be enough if we require the ITE for optical metrology measurements, fulfilling conditions (14)-(15).

Acknowledgments

1. M.R. Teague, *J. Opt. Soc. Am.* **73** (1983) 1434.

2. N. Streibl, *Opt. Commun.* **49** (1983) 6.

3. K. Ichikawa, A.W. Lohmann, and M. Takeda, *Appl. Opt.* **27** (1988) 3433.

4. P. Alonso-Magaña, F.S. Granados-Agustin, and A. Cornejo-Rodriguez, *Rev. Mex. Fís.* **46 S2** (2000) 54.

5. F. Roddier, *Appl. Opt.* **29** (1990) 1402.

6. D. Malacara, M. Servín, and Z. Malacara, *Interferogram analysis for optical testing* (Marcel Dekker, Inc., New York, 1998) p. 366.

7. M. Fernández-Guasti, J. L. Jiménez, F. Granados-Agustín, and A. Cornejo-Rodríguez, *J. Opt. Soc. Am. A* **20** (2003) 1629.

8. E. Kolenovic, *J. Opt. Soc. Am. A* **22** (2005) 899.

9. M. Campos-García, *Prueba de Roddier: Una revisión*, B. Sc. Thesis (Universidad Nacional Autónoma de México, Distrito Federal, México, 1995) p. 15.

10. J.D. Jackson, *Classical electrodynamics*, 2nd ed. (John Wiley & Sons, Inc., USA, 1975) p. 236.

11. M. Born and E. Wolf, *Principles of optics*, 7th (expanded) ed. (Cambridge University Press, Cambridge, 1999) p. 116.

12. J.W. Goodman, *Introduction to Fourier optics*, 2nd ed. (McGraw-Hill, New York, 1996) p. 402.

13. A. Cornejo-Rodriguez and A. Cordero-Davila, "Wavefront slope measurements in optical testing", in *Handbook of Optical Design*, D. Malacara, B.J. Thompson ed., (Marcel Dekker, Inc., New York, 2001) p. 311.

14. D. Malacara and S.L. Devore, "Interferogram evaluation and wavefront fitting", in *Optical Shop Testing*, 2nd ed., D. Malacara, ed., (Wiley, New York, 1992) p. 455.